Improved Bounds on the List Decreasing Heuristic for the Vertex Cover Problem

Tai-Pao Chuang
Dept. of CSIE, National Taiwan Normal University and Ching Yun University
Email: tpchuang@cyu.edu.tw

Shun-Shii Lin
Dept. of CSIE, National Taiwan Normal University
Email: linss@csie.ntnu.edu.tw

Abstract—The list decreasing heuristic for the vertex cover problem is an online vertex covering algorithm. An upper bound $\sqrt{\Delta}/2+3/2$ had been proposed on the approximation ratio for it, and a graph had also been given to reach a lower bound of $\sqrt{\Delta}/2+1/2$. In this paper, we refine the techniques of previous researchers, and construct a new type of graphs which can enhance the lower bound to $\sqrt{\Delta}/2+1$, and all the graphs can be categorized into a group. Then we proposed a modified algorithm to obtain a tighter bound and prove it with an example.

Index Terms—approximation algorithm, list decreasing heuristic, vertex cover problem.

I. INTRODUCTION

The minimum vertex cover problem is the optimization problem of finding a minimum cardinality vertex cover for a given graph. Let $G = (V, E)$ be an undirected, unweighted graph. A set of vertices $C \subseteq V$ is called a vertex cover if for any edge in E at least one of its endpoints is contained in C.

The vertex cover problem is a famous NP-hard optimization problem. Currently, there is no polynomial time algorithm to solve it optimally. Several approximation algorithms for the vertex cover problem have been proposed with various performance guarantees. Cormen et al. [1] described a very simple approximation algorithm based on maximal matching which gives an approximation ratio of 2.

Demange et al. [2] proposed the online vertex covering problem. The input is not entirely known at the beginning, vertices are revealed one by one and a decision of selection must be taken for each revealed vertex. The scanned vertex is selected if and only if it has at least a nonselected already revealed neighbor.

Some approximation algorithms are based on a static ordering of vertices determined by their degrees and the vertex degrees are not updated in the process. An interesting model is what Avis et al. called the list heuristic in [3]. This kind of algorithm scans the vertices one by one in a fixed given order called a list and takes a definitive decision of selection for the currently scanned vertex. Avis et al. [3] proposed the list decreasing heuristic (Delbot called it ListLeft [4]) that chooses vertices in order of decreasing degree, selecting a vertex if it is adjacent to an uncovered edge. They proved that its approximation ratio is at most

$$\frac{\Delta}{2\sqrt{\Delta}-1}+1 \le \frac{\sqrt{\Delta}}{2}+\frac{3}{2},$$

where Δ is the maximum degree of the graph. They also showed that a lower bound on the approximation ratio is $\sqrt{\Delta}/2+1/2$.

Since less information is available at each step, the list decreasing heuristic ListLeft performs worse than the greedy algorithm which repeatedly selects a vertex adjacent to the largest number of uncovered edges. On the other hand, Delbot et al. [4] introduced a better list heuristic algorithm ListRight which treats vertices in increasing order of their degrees.

In this paper, we obtain a tighter upper bound

for the list decreasing heuristic in [3]. Moreover, we construct a group of graphs to show that the list decreasing heuristic has a lower bound of $\sqrt{\Delta}/2+1$. The results is better than that of Avis et al. [3], which is $\sqrt{\Delta}/2+1/2$。

This paper is organized as follows. Section 2 briefly reviews the terminologies. Section 3 proposes a tighter upper bound for the list decreasing heuristic algorithm. Section 4 constructs a group of graphs in recursion form. Lastly a brief conclusion is made in Section 5.

II. PRELIMINARIES

A. Vertex cover problem

A vertex cover is a set of vertices in a graph. Given an undirected graph $G = (V, E)$, a set of vertices $C \subseteq V$ is called a vertex cover if for any edge in E at least one of its endpoints is contained in C. In other words, given G, a vertex cover of G is a set of vertices V' such that $V' \subseteq V$, $\forall (u, v) \in E, u \in V'$ or $v \in V'$ or both.

For any vertex $v_i \in V$, we denote $N(v_i)$ the set of neighbors of v_i and $d_i = |N(v_i)| = d(v_i)$ the degree of v_i, i.e. its number of neighbors. Let n represent the number of vertices, m the number of edges, and $\triangle$ the maximum degree of G. Assume the vertices are labeled such that $\Delta = d_1 \geq d_2 \geq \ldots \geq d_n$. Let C_D be a vertex cover and opt denote the size of the minimum vertex cover, and then $|C_D|/opt$ denotes the approximation ratio.

B. List heuristic algorithm

Demange et al. [2] proposed an online vertex covering algorithm. The scanned vertex is selected if and only if it has at least a nonselected already revealed neighbor. List heuristic algorithm is a kind of online vertex covering algorithm. Any permutation of the n vertices of V is called a list. The list may be sorted according to the vertices′ degrees.

The vertices are revealed one by one from the list. The algorithm scans the list in real time and makes a decision of selection or not for the currently scanned vertex immediately. The vertex degrees are not updated during the process.

List decreasing heuristic algorithm was presented by Avis et al. [3], in 2007, which is called ListLeft in [4]. It scans vertices in order of decreasing degree (from left to right) and selects a vertex if it is adjacent to an uncovered edge. Delbot et al. [4] modified the list decreasing heuristic algorithm to ListRight heuristic algorithm. They scan the list from right to left. The scanned vertex is selected if and only if at least a right neighbor is not selected.

In this paper, we deal with ListLeft and analyze its bounds on the approximation ratio.

C. Linear programming

An important tool in the analysis of approximation algorithms is a linear programming relaxation of the related integer programming problem, and was first used by Lovász [3]. The traditional approach is shown in Figure 1.
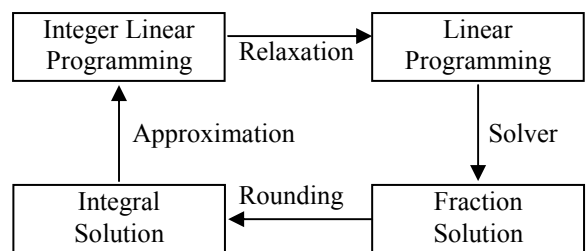

Figure 1. The traditional approach in the integer linear programming problem.

The following definition is a linear programming relaxation for the vertex cover problem.

For any vertex cover C of a graph $G = (V, E)$, we denote x_v the weight of vertex v, and define $X=\{x_v\}_{v \in V}$ as follows：$x_v = \begin{cases} 1, & v \in C, \\ 0, & v \notin C, \end{cases}$

where X is a 0 or 1 valued feasible solution for the linear programming, and conversely every 0 or 1 feasible solution corresponds to a vertex cover for G.

For any edge $e=uv \in E$ of a graph $G = (V, E)$, we denote y_e the weight of edge e, and define $Y=\{y_e\}_{e \in E}$ as follows：$y_e = 1/d(u)$ or $1/d(v)$ or $1/\Delta$ depending on the designed algorithm, where Y is a fraction valued feasible solution for the linear programming. This is an assignment of nonnegative weights to the edges of G such that the sum of the edge weights at any vertex is at most one. Any $Y=\{y_e\}_{e \in E}$ that satisfies the conditions above is called a fractional matching of G.

We define the size of X as $|X| = \sum_{v \in V} x_v$, and clearly $|X|=|C|$. We define the size of Y as $|Y| = \sum_{e \in E} y_e$. In the vertex cover problem, we try to minimize the summation of x_v, where $v \in V$.

$$\min \sum_{v \in V} x_v$$
$$s.t. \quad x_v + x_u \geq 1 \quad \forall (u,v) \in E,$$
$$x_v \geq 0 \quad \forall v \in V.$$

The dual is to maximize the summation of y_e, where $e \in E$.

$$\max \sum_{e \in E} y_e$$
$$s.t. \quad \sum_{e \in \delta(v)} y_e \leq 1 \quad \forall v \in V,$$
$$y_e \geq 0 \quad \forall e \in E.$$

For any graph G, let opt be an minimum vertex cover of G, C_D be any vertex cover of G and Y be any fractional matching. Clearly

$$|C_D| = |X| \geq opt \geq |Y| = \sum_{e \in E} y_e.$$

Using the above facts, Avis et al. obtained some bounds on the approximation ratio for the ListLeft algorithm.

III. AN ANALYSIS FOR THE UPPER BOUND ON THE APPROXIMATION RATIO

Given a graph $G = (V, E)$, let d_i denote the degree of vertex v_i, and assume the vertices are labeled such that $\Delta = d_1 \geq d_2 \geq \ldots \geq d_n$. The maximum degree is $\triangle = d_1$ and the minimum degree is d_n. The list decreasing heuristic algorithm [3] is shown below.

Algorithm ListLeft.

Input：Any graph G and any associated list $L=<v_1,v_2,\ldots,v_n>$ sorted by decreasing degrees, i.e. $\Delta = d_1 \geq d_2 \geq \ldots \geq d_n$.

$C_D := \varnothing;$ // Initially C_D is empty.

For i=1 to n-1 //Scan the list L from left to right.

{

  Let v_i be the currently scanned vertex;

  Let R_i denote the set of edges incident to v_i but not incident to any vertex already in C_D;

  If R_i is not empty, then $C_D := C_D \cup \{v_i\};$

};

Return $(C_D);$

        □

The above algorithm outputs a vertex cover C_D in n-1 steps by scanning the vertices one by one. In [3], Avis et al. constructed a dual feasible solution $Y=\{y_e\}_{e \in E}$ as follows. Let p be the minimum index such that $\sum_{i=1}^{p} d_i \geq m$. $\{v_1,v_2,\ldots,v_p\}$ is a minimum cardinality set of vertices. Y is obtained by initially setting $y_e = \frac{1}{\Delta}$ for each edge e. For each $i \geq p+1$ for which v_i is selected by ListLeft, they choose an arbitrary edge e from R_i and reassign it a weight

$y_e = \dfrac{1}{d_i}$. Now a fractional matching Y=$\{y_e\}_{e \in E}$ of G can be obtained from C_D.

In the following, we will refine the techniques of Avis et al. [3] and Iida Hiroshi [5, 6] to derive the following theorem.

Theorem 1.

Let C_D be the solution obtained by ListLeft and opt be the size of the optimal solution, then

$$|C_D| \le opt * \frac{\Delta}{\sqrt{2\Delta}-1} + \frac{(\sqrt{2\Delta}+1)}{4 + \dfrac{2+4d_n}{\Delta-1-d_n}}.$$

Proof.

In the selected part C_D obtained by ListLeft, we choose the minimum p nodes $v_1, v_2, \ldots, v_p$ such that the degree sum of the p nodes is just greater than or equal to m, i. e. $\sum_{i=1}^{p} d_i \ge m$. We then let

$s = |C_D| - p$.

First, we focus on the first p nodes. The degree sum of the p nodes will be less than or equal to $m-1+\triangle$ as shown below.

$\because d_p \ge 1$

$\therefore d_1 + d_2 + \ldots + d_{p-1} \le m-1$

$\because d_p \le \Delta$

$\therefore d_1 + d_2 + \ldots + d_{p-1} + d_p \le m-1+\Delta$

Second, we focus on the following s nodes. The degree sum of the s nodes will be less than or equal to $m-d_n$ as below.

$\because \sum_{i=1}^{p} d_i \ge m$ and $\sum_{i=1}^{n} d_i = \sum_{i=1}^{p} d_i + \sum_{i=p+1}^{p+s} d_i + \sum_{i=p+s+1}^{n} d_i = 2m$,

$\therefore d_{p+1} + \ldots + d_{p+s} \le m-d_n$.

We then apply the Cauch-Schwarz inequality used in [3]. Since $d_{p+1}, d_{p+2}, \ldots, d_{p+s}$ are positive integers with sum at most m- d_n, then

$$\sum_{i=p+1}^{p+s} \frac{1}{d_i} \ge \frac{s^2}{m-d_n}.$$

Let opt be the size of the optimal solution. Now we construct a different fractional matching Y=$\{y_e\}_{e \in E}$ as follows. Let R_i denote the set of edges incident to v_i but not incident to any vertex already in C_D. We select an arbitrary edge e from R_i and assign it the weight $y_e = \dfrac{1}{d_i}$. Each of the other edges in E is assigned the weight $\dfrac{1}{\Delta} = \dfrac{1}{d_1}$. Then

$$opt \ge \sum y_e = \sum_{i=1}^{p} \frac{1}{d_i} + \sum_{i=p+1}^{p+s} \frac{1}{d_i} + \frac{m-|C_D|}{d_1}$$

$$\ge \frac{p^2}{m-1+\Delta} + \frac{s^2}{m-d_n} + \frac{m-|C_D|}{d_1}$$

$$= \left[\frac{p^2}{m-1+\Delta} + \frac{m-1+\Delta}{2d_1}\right] + \left[\frac{s^2}{m-d_n} + \frac{m-d_n}{2d_1}\right] + 2\left[\frac{m-|C_D|}{d_1}\right]$$

$$- \frac{m-1+\Delta}{2d_1} - \frac{m-d_n}{2d_1}$$

Using the arithmetic-geometric mean inequality,

$$opt \ge 2\sqrt{\frac{p^2}{2d_1}} + 2\sqrt{\frac{s^2}{2d_1}} + \frac{2m-2|C_D|-(m-1+\Delta)-(m-d_n)}{2d_1}$$

$$= \sqrt{\frac{2}{d_1}}(p+s) - \frac{|C_D|}{d_1} + \frac{1-\Delta+d_n}{2d_1}$$

$$= |C_D|\left[\sqrt{\frac{2}{d_1}} - \frac{1}{d_1}\right] + \frac{1-\Delta+d_n}{2d_1}$$

$$opt - \frac{1-\Delta+d_n}{2\Delta} \ge (|C_D|)(\sqrt{\frac{2}{\Delta}} - \frac{1}{\Delta}),$$

$$|C_D| \le opt * \frac{\Delta}{\sqrt{2\Delta}-1} + \frac{\Delta-1-d_n}{2\sqrt{2\Delta}-2}$$

$$\le opt * \frac{\Delta}{\sqrt{2\Delta}-1} + \frac{(\sqrt{2\Delta}+1)}{4 + \dfrac{2+4d_n}{\Delta-1-d_n}}. \tag{1}$$

$\square$

The equation (1) is more precise than the result in [5], but not good enough. So we propose a tighter upper bound for ListLeft in the following theo-

rem.

Theorem 2.

Let C_D be the solution obtained by ListLeft, k be the number of vertices with maximum degree, and opt be the size of the optimal solution. If $k < n$, then

$$\frac{|C_D|}{opt} \leq \frac{n-1}{\Sigma y_e} = \frac{n-1}{\sum\limits_{i=1}^{n-1} \frac{1}{d_i} + \frac{m-(n-1)}{d_1}} \leq \frac{n-1}{\frac{(n-1-k)^2}{2m-k\Delta-d_n} + \frac{m-n+1+k}{\Delta}}.$$

Proof.

In ListLeft, we divide the n nodes into two parts, the selected part and unselected part. The selected part is equal to the vertex cover C_D. In the selected part we have at most n-1 nodes and remain at least one node to be unselected. We can write the equation as follow.

$$|C_D| \leq n-1. \tag{2}$$

Now we construct a fractional matching $Y = \{\frac{1}{d_1}, \frac{1}{d_2}, ..., \frac{1}{d_{n-1}}, \frac{1}{d_1}, ..., \frac{1}{d_1}\}$. The total weight of the m edges can be written as follow.

$$\Sigma y_e = \frac{1}{d_1} + \frac{1}{d_2} + ... + \frac{1}{d_{n-1}} + \frac{m-(n-1)}{d_1} = \sum\limits_{i=1}^{n-1} \frac{1}{d_i} + \frac{m-(n-1)}{d_1}.$$

$\because opt \geq \Sigma y_e.$

$$\therefore opt \geq \sum\limits_{i=1}^{n-1} \frac{1}{d_i} + \frac{m-(n-1)}{d_1}. \tag{3}$$

$\because d_1 + d_2 + ... + d_{n-1} + d_n = 2m.$

If $d_1 = d_2 = ... = d_k = \Delta$ and $k < n$, then

$d_{k+1} + ... + d_{n-1} = 2m - k\Delta - d_n.$

We use the techniques proposed by Avis et al. [3] to reduce the formula as follows :

$$\frac{1}{d_{k+1}} + ... + \frac{1}{d_{n-1}} \geq \frac{(n-1-k)^2}{2m-k\Delta-d_n}.$$

$$opt \geq \Sigma y_e = \frac{1}{d_1} + \frac{1}{d_2} + ... + \frac{1}{d_{n-1}} + \frac{m-(n-1)}{d_1}$$

$$\geq \frac{(n-1-k)^2}{2m-k\Delta-d_n} + \frac{m-(n-1-k)}{\Delta}$$

$$= \frac{(n-1-k)^2}{2m-k\Delta-d_n} + \frac{m-n+1+k}{\Delta}.$$

$$\frac{1}{opt} \leq \frac{1}{\frac{(n-1-k)^2}{2m-k\Delta-d_n} + \frac{m-n+1+k}{\Delta}}. \tag{4}$$

From (2) and (4), we can get the approximation ratio as follows :

$$\frac{|C_D|}{opt} \leq \frac{n-1}{\Sigma y_e} = \frac{n-1}{\sum\limits_{i=1}^{n-1} \frac{1}{d_i} + \frac{m-(n-1)}{d_1}} \leq \frac{n-1}{\frac{(n-1-k)^2}{2m-k\Delta-d_n} + \frac{m-n+1+k}{\Delta}}. \tag{5}$$

$\square$

Let us show an example in Figure 2. Note that the total number of vertices n=25, the total number of edges m=84, the maximum degree Δ=16 =N^2, the number of vertices with maximum degree k=4. This example can be generalized to let N be an arbitrary value.

We have

$$opt \geq \frac{(n-1-k)^2}{2m-k\Delta-d_n} + \frac{m-n+1+k}{\Delta}$$

$$= \frac{(25-1-4)^2}{2*84-4*16-4} + \frac{84-25+1+4}{16} = 8,$$

$|C_D| \leq n-1 = 24,$

$$\frac{|C_D|}{opt} \leq \frac{24}{8} = 3.$$

Now we find the lower bound formula in [3], we can arise to $\frac{\sqrt{\Delta}}{2} + 1 = \frac{\sqrt{16}}{2} + 1 = 3$.

We can find that the upper bound on the approximation ratio using (5) is just equal to the lower bound $\sqrt{\Delta}/2+1$ in [3]. This means we have found an example to close the gap between the two bounds.

Let us deal with the case that all vertices in the graph have the maximum degree.
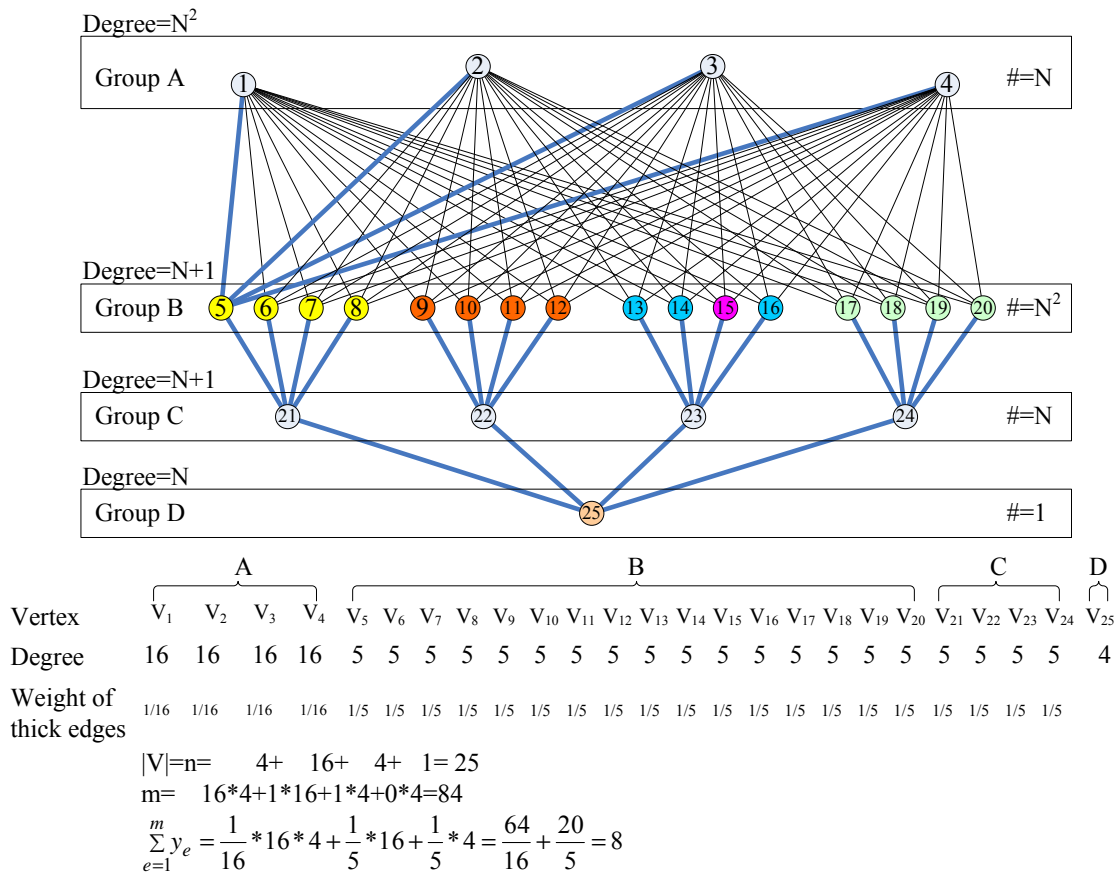

		A											B									C			D
Vertex	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9	V_{10}	V_{11}	V_{12}	V_{13}	V_{14}	V_{15}	V_{16}	V_{17}	V_{18}	V_{19}	V_{20}	V_{21}	V_{22}	V_{23} V_{24}	V_{25}
Degree	16	16	16	16	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5 5	4
Weight of thick edges	1/16	1/16	1/16	1/16	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5 1/5	

$|V| = n = \quad 4 + \quad 16 + \quad 4 + \quad 1 = 25$

$m = \quad 16*4 + 1*16 + 1*4 + 0*4 = 84$

$$\sum_{e=1}^{m} y_e = \frac{1}{16} * 16 * 4 + \frac{1}{5} * 16 + \frac{1}{5} * 4 = \frac{64}{16} + \frac{20}{5} = 8$$

Figure 2. An example using *N*=4.

Corollary 1.

Consider a graph $G=(V, E)$ with $k = n$ and $d_1 = d_2 = \ldots = d_n = \Delta$. Let C_D be the solution obtained by ListLeft applied on G and *opt* be the size of the optimal solution, then $\frac{|C_D|}{opt} \leq \frac{n-1}{\sum y_e} = \frac{(n-1)\Delta}{m}$.

Proof.

If $d_1 = d_2 = \ldots = d_n = \Delta$ and $k = n$, then each edge $e \in E$ can be assigned the weight $y_e = 1/\Delta$, and $opt \geq \sum y_e = \frac{m}{\Delta}$. From (2), we can get the approximation ratio as follows：

$$\frac{|C_D|}{opt} \leq \frac{n-1}{\sum y_e} = \frac{n-1}{\frac{m}{\Delta}} = \frac{(n-1)\Delta}{m} \qquad (6)$$

$\square$

Example.

Let us show the results of complete graphs in Figure 3. In complete graphs, if we use maximum degree to express the approximation ratio then we will get bigger ratio when *n* is bigger. So we can get a tighter bound,

$$\frac{|C_D|}{opt} \leq \frac{(n-1)\Delta}{m} = \frac{(n-1)(n-2)}{n(n-2)/2} = 2 - \frac{2}{n}$$

in our formula than that of Avis et al. [3].

The exact approximation ratio in the complete graphs is always $\dfrac{|C_D|}{opt} \le (n\text{-}1)/(n\text{-}1) = 1$.
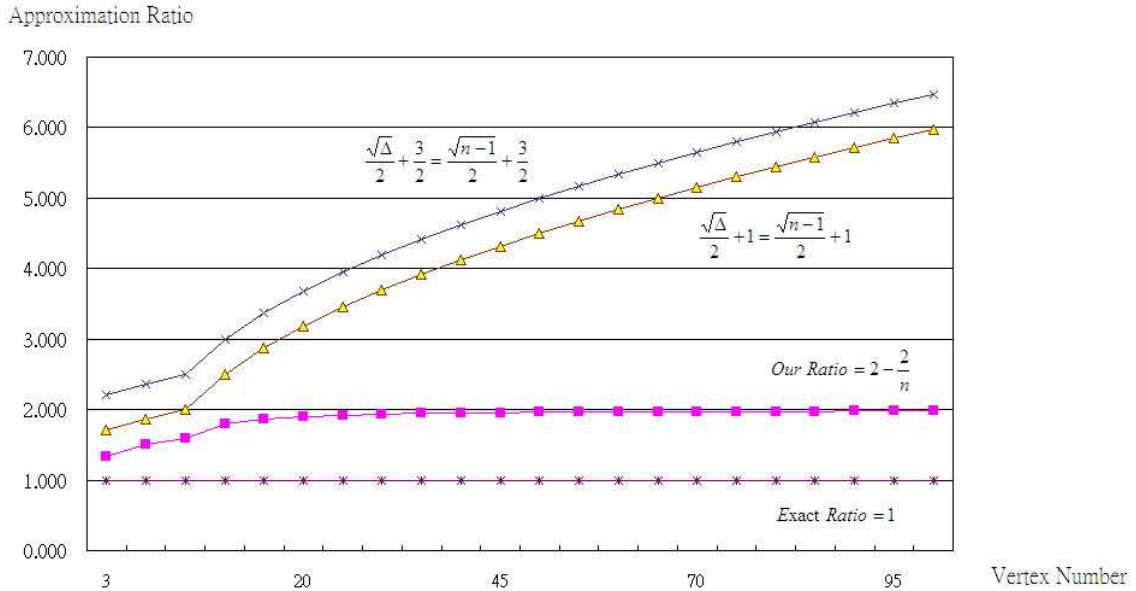

Figure 3. The result of complete graphs using n=3 to 100.

Let us deal with the case that the graph has t vertices with minimum degree 1 or 0.

Corollary 2.

Consider a graph $G=(V, E)$ with $d_1 = d_2 = \ldots = d_k = \Delta$, $k < n$, and $d_{n-t+1} = d_{n-t+2} = \ldots = d_n = 1$ or 0, i.e., there are t vertices with minimum degree 1 or 0. Let C_D be the solution obtained by ListLeft applied on G and opt be the size of the optimal solution, then

$$\frac{|C_D|}{opt} \le \frac{n-t}{\Sigma y_e} = \frac{n-t}{\sum\limits_{i=1}^{n-q}\frac{1}{d_i} + \frac{m-(n-t)}{d_1}} \le \frac{n-t}{\frac{(n\text{-}t\text{-}k)^2}{2m\text{-}k\Delta\text{-}t} + \frac{m-n+t+k}{\Delta}}.$$

Proof.

In ListLeft, we divide the n nodes into two parts, the selected part and unselected part. The selected part is equal to the vertex cover C_D. If there are t vertices with minimum degree 1 or 0, then all the t vertices will be put into the unselected part. So we have

$|C_D| \le n - t$.   (7)

The weights of the edges can be assigned similarly to those in the proof of Theorem 2.

We construct a fractional matching Y = $\{\frac{1}{d_1}, \frac{1}{d_2}, \ldots, \frac{1}{d_{n-1}}, \frac{1}{d_1}, \ldots, \frac{1}{d_1}\}$. Then

$$\Sigma y_e = \frac{1}{d_1} + \frac{1}{d_2} + \ldots + \frac{1}{d_{n-t}} + \frac{m-(n-t)}{d_1} == \sum\limits_{i=1}^{n-t}\frac{1}{d_i} + \frac{m-(n-t)}{d_1}.$$

$$\Sigma y_e = \frac{1}{d_1} + \frac{1}{d_2} + \ldots + \frac{1}{d_{n-t}} + \frac{m-(n-t)}{d_1}$$
$$= \sum\limits_{i=1}^{n-t}\frac{1}{d_i} + \frac{m-(n-t)}{d_1}.$$

$\because opt \ge \Sigma y_e.$

$\therefore opt \ge \sum\limits_{i=1}^{n-t}\frac{1}{d_i} + \frac{m-(n-t)}{d_1}.$

$d_1 + d_2 + \ldots + d_{n-1} + d_n = 2m.$

If $d_1 = d_2 = \ldots = d_k = \Delta$, $d_{n-t+1} = d_{n-t+2} = \ldots = d_n = 1$ or 0, and k < n, then

$d_{k+1} + \ldots + d_{n-t} = 2m - k\Delta - t.$

We use the techniques proposed by Avis et al. [3] to reduce the formula as follows：

$$\frac{1}{d_{k+1}} + \ldots + \frac{1}{d_{n-t}} \geq \frac{(n-t-k)^2}{2m-k\Delta-t}.$$

$$opt \geq \Sigma y_e = \frac{1}{d_1} + \frac{1}{d_2} + \ldots + \frac{1}{d_{n-t}} + \frac{m-(n-t)}{d_1}$$

$$\geq \frac{(n-t-k)^2}{2m-k\Delta-t} + \frac{m-(n-t-k)}{\Delta}$$

$$= \frac{(n-t-k)^2}{2m-k\Delta-t} + \frac{m-n+t+k}{\Delta}.$$

$$\frac{1}{opt} \leq \frac{1}{\frac{(n-t-k)^2}{2m-k\Delta-t} + \frac{m-n+t+k}{\Delta}}. \qquad (8)$$

From (8) and (9), we can get the approximation ratio as follows：

$$\frac{|C_D|}{opt} \leq \frac{n-t}{\Sigma y_e} = \frac{n-t}{\sum_{i=1}^{n-t}\frac{1}{d_i} + \frac{m-(n-t)}{d_1}} \leq \frac{n-t}{\frac{(n-t-k)^2}{2m-k\Delta-t} + \frac{m-n+t+k}{\Delta}}. \quad (9)$$

□

Example.

Let us use the bipartite graph as an example in Figure 4. Note that the total number of vertices $n=17$, the total number of edges $m=18$, the maximum degree $\Delta=3$, the number of vertices with maximum degree $k=8$, the number of vertices with minimum degree $t=6$. In this bipartite graph,

$$|C_D| \leq n-t = 17-6 = 11.$$

Now we assign the value $1/\Delta = 1/3$ as the weight of the thick edges in Figure 4. The other edges are also assigned a weight of $1/\Delta = 1/3$.

Case 1 : Using Equation (9), we have

$$opt \geq \Sigma y_e = \frac{(n-t-k)^2}{2m-k\Delta-t} + \frac{m-n+t+k}{\Delta} = 6.5,$$

$$\frac{|C_D|}{opt} \leq \frac{n-t}{\Sigma y_e} = 11/6.5 = 1.692.$$

Case 2 : Using the upper bound in [3],

$$\frac{|C_D|}{opt} \leq \frac{\Delta}{2\sqrt{\Delta}-1} + 1 = \frac{3}{2\sqrt{3}-1} + 1 = 2.2175.$$

Case 3 : Using the upper bound we proposed,

$$\frac{|C_D|}{opt} \leq \frac{\sqrt{\Delta}}{2} + 1 = \frac{\sqrt{3}}{2} + 1 = 1.866.$$

We can find $1.866 < 2.2175$. We have a tighter bound $\sqrt{\Delta}/2 + 1$ than those of Avis et al. [3].
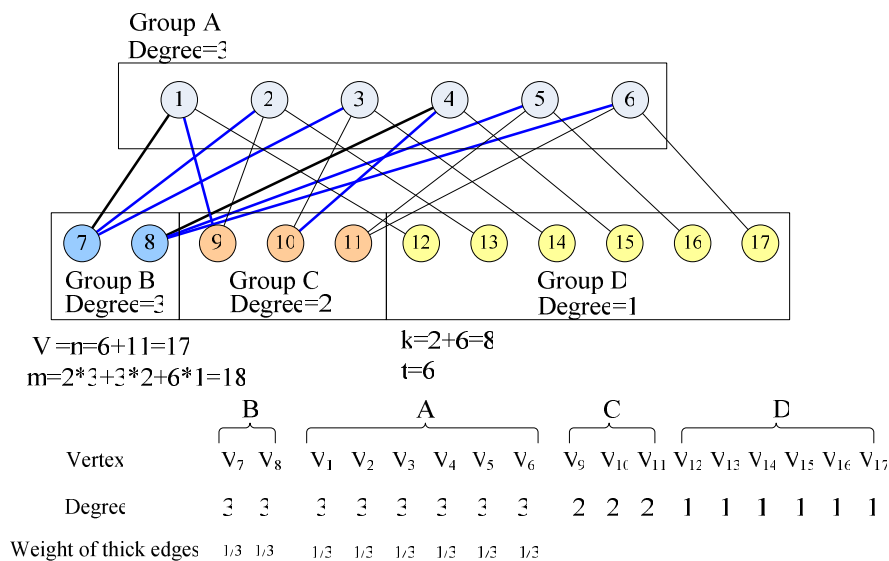
Figure 4. An example using maximum degree $\Delta=3$.

		B		A						C			D					
Vertex		V_7	V_8	V_1	V_2	V_3	V_4	V_5	V_6	V_9	V_{10}	V_{11}	V_{12}	V_{13}	V_{14}	V_{15}	V_{16}	V_{17}
Degree		3	3	3	3	3	3	3	3	2	2	2	1	1	1	1	1	1
Weight of thick edges		1/3	1/3	1/3	1/3	1/3	1/3	1/3	1/3									

IV. A NEW GROUP OF GRAPHS IN RECURSIVE FORM TO MATCH THE LOWER BOUND

Reducing the gap between the lower bound and upper bound is one of the main challenges facing the researchers. In this section, we propose a new group of graphs which can enhance the lower bound on the approximation ratio for ListLeft proposed by Avis et al. [3].

The graphs can be represented as a recursion structure shown in Figure 5. The maximum degree of the graph is N^{2k}, so $\sqrt{\Delta} = N^k$, then the lower bound on the approximation ratio will be $\frac{N^{2k} + 2N^k}{2N^k} = \frac{N^k}{2} + 1$. It means that the approximation ratio of the lower bound is also come up to $\sqrt{\Delta}/2 + 1$.

Avis et al. [3] obtained an upper bound $\sqrt{\Delta}/2 + 3/2$ which can be represented in more precise form as $\frac{\Delta}{2\sqrt{\Delta} - 1} + 1$. If we let $\sqrt{\Delta} = N^k$, then the upper bound in [3] will be $\frac{N^{2k} + 2N^k - 1}{2N^k - 1}$. The approximation ratio of our graphs can be written as $\frac{N^{2k} + 2N^k}{2N^k}$. The gap between the two bounds is

$$\frac{N^{2k} + 2N^k - 1}{2N^k - 1} - \frac{N^{2k} + 2N^k}{2N^k} = \frac{N^{2k}}{4N^{2k} - 2N^k} = \frac{1}{4 - 2/N^k}$$

and $\lim_{N \to \infty} \frac{1}{4 - 2/N^k} = 0.25$. It means the gap between the upper and lower bounds is come up to 0.25 when n tends infinity.

Example.

In Figure 6, the maximum degree of the graph is N^2, so $\sqrt{\Delta} = N$, then its lower bound on the approximation ratio will be $\frac{N^2 + 2N}{2N} = \frac{N}{2} + 1$. It means that the approximation ratio of the lower bound is come up to $\sqrt{\Delta}/2 + 1$.

Group	Degree	Total Nodes	=	#	*	Set
A	N^{2k}	N^k	=	N	*	N^{k-1}
B	N^k+1	N^{2k}	=	N^2	*	N^{2k-2}
C	N^k+1	N^k	=	1	*	N^k
D	N^k	1	=	1	*	1

$|C_D| = N^k + N^{2k} + N^k = N^{2k} + 2N^k$
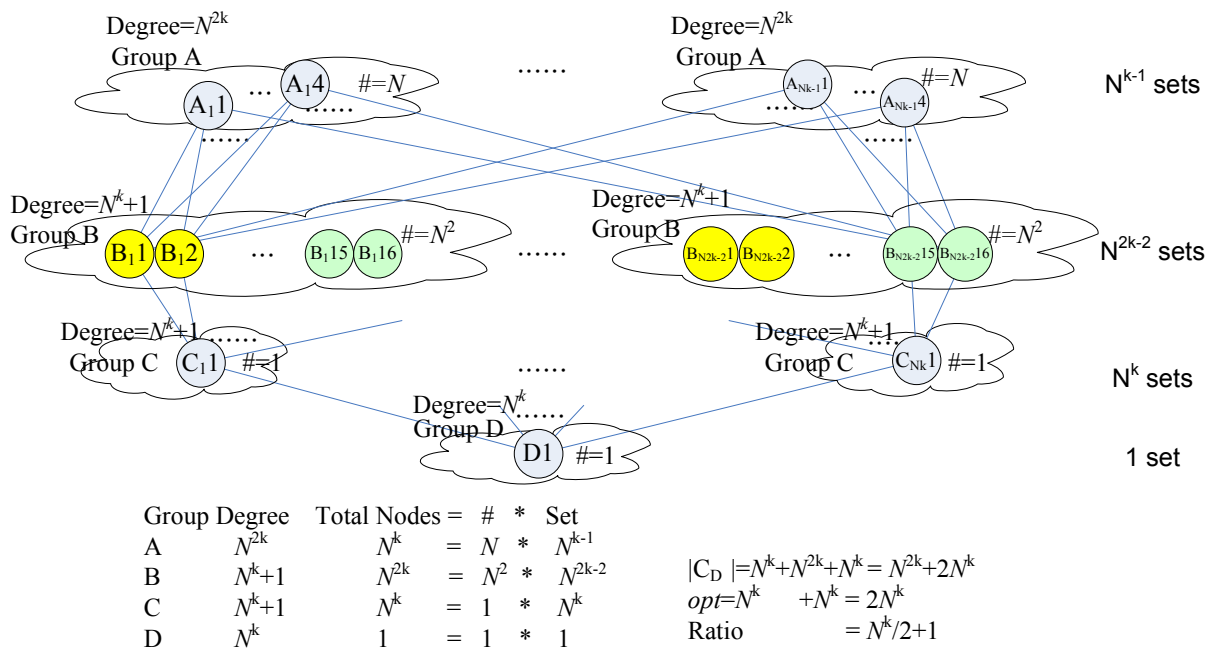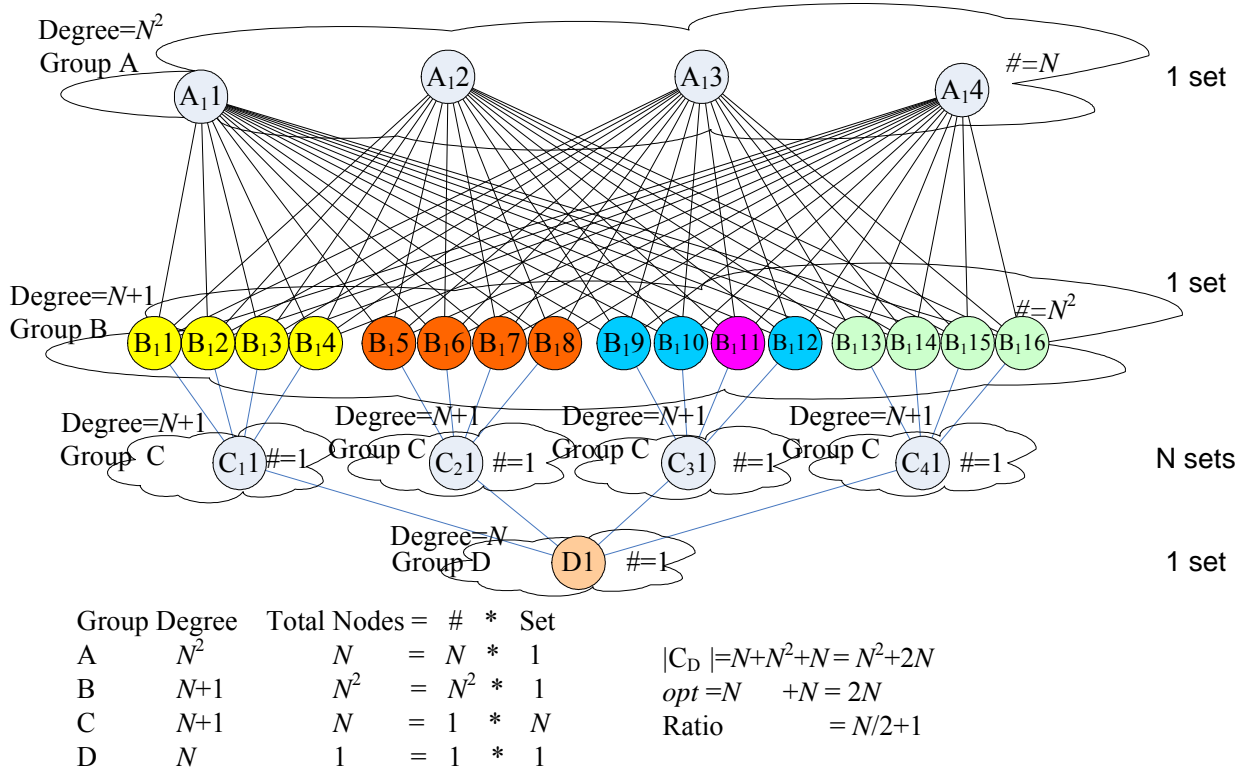
$opt = N^k + N^k = 2N^k$

Ratio $= N^k/2 + 1$

Figure 5. The structure of the example groups.

Figure 6. An example of graphs with maximum degree N^2

IV. CONCLUSION

After careful consideration and derivation, we obtain the equation (1) in Theorem 1. The result is better than that of Iida Hiroshi [5, 6]. Furthermore, we have proposed a new group of graphs which can come up the lower bound to $\sqrt{\Delta}/2+1$. The result is also better than that of Avis et al. [3]. The gap between the lower bound and upper bound is come up to 0.25 when n tends infinity.

We have proposed a tighter bound formula of the list decreasing heuristic for the vertex cover problem. We also give an example to show the result of the upper bound is equal to $\sqrt{\Delta}/2+1$. The bounds of ListLeft derived by some researchers and us are shown in Figure 7.

In [3] the authors proved that an upper bound on the approximation ratio of ListLeft is

$\sqrt{\Delta}/2+3/2$, and it can be represented in more precise form as $\dfrac{\Delta}{2\sqrt{\Delta}-1}+1$. The difference between the two bounds and the exact upper bound is $\dfrac{1}{4}-\dfrac{\sqrt{\Delta}+1/2}{8\Delta-2}$. Iida Hiroshi [5] used the lemma proposed by Avis et al. [3] to obtain the following formula：

$$|C_D| \le opt * \frac{\Delta}{\sqrt{2\Delta}-1} + \frac{\sqrt{2\Delta}+1}{4}.\tag{10}$$

Furthermore Iida Hiroshi [6] got the following equation:

$$|C_D| \le opt * \frac{\Delta}{\sqrt{2\Delta}-1} - \frac{q}{2\sqrt{2\Delta}-2}, \text{ where } q \ge d_n.\tag{11}$$

Iida Hiroshi [6] indicated that, when the maximum degree (Δ) is smaller than 19, the above upper bound is better than that of Avis′s. Finally,

we summarize our and previous results in Table 1.

∞

$$|C_D| \leq opt * (\frac{\sqrt{\Delta}}{2} + \frac{3}{2}) \quad \text{in [3].}$$

$$|C_D| \leq opt * \frac{\Delta}{\sqrt{2\Delta} - 1} + \frac{\sqrt{2\Delta} + 1}{4} \quad \text{in [6],}$$
when $\Delta \geq 19$ the result is worse than $|C_D| \leq opt * (\frac{\Delta}{2\sqrt{\Delta} - 1} + 1)$ in [3].

We propose an upper bound (in Theorem 1)
$$|C_D| \leq opt * \frac{\Delta}{\sqrt{2\Delta} - 1} + \frac{(\sqrt{2\Delta} + 1)}{4 + \frac{2 + 4d_n}{\Delta - 1 - d_n}}$$
when $\Delta \geq 19$ the result is worse than $|C_D| \leq opt * (\frac{\Delta}{2\sqrt{\Delta} - 1} + 1)$ in [3].

$\frac{1}{4} - \frac{\sqrt{\Delta} + \frac{1}{2}}{8\Delta - 2}$

A more precise form $|C_D| \leq opt * (\frac{\Delta}{2\sqrt{\Delta} - 1} + 1)$ in [3].

$$|C_D| \leq opt * \frac{\Delta}{\sqrt{2\Delta} - 1} + \frac{\sqrt{2\Delta} + 1}{4} \quad \text{in [6],}$$
when $\Delta < 19$ the result is better than $|C_D| \leq opt * (\frac{\Delta}{2\sqrt{\Delta} - 1} + 1)$ in [3].

$\frac{1}{4 - 2/\sqrt{\Delta}}$

We propose an upper bound (in Theorem 1)
$$|C_D| \leq opt * \frac{\Delta}{\sqrt{2\Delta} - 1} + \frac{(\sqrt{2\Delta} + 1)}{4 + \frac{2 + 4d_n}{\Delta - 1 - d_n}}$$
when $\Delta < 19$ the result is better than $|C_D| \leq opt * (\frac{\Delta}{2\sqrt{\Delta} - 1} + 1)$ in [3].

We propose an upper bound (in Theorem 2)
$$|C_D| \leq opt * \frac{n - 1}{\frac{(n - 1 - k)^2}{2m - k\Delta - d_n} + \frac{m - n + 1 + k}{\Delta}}.$$

We propose a new group of graphs (in Section 4) which enhances the lower bound $|C_D| \geq opt * (\frac{\sqrt{\Delta}}{2} + 1)$.

0.5

Avis et al. [3] proposed a graph to show the lower bound is at least
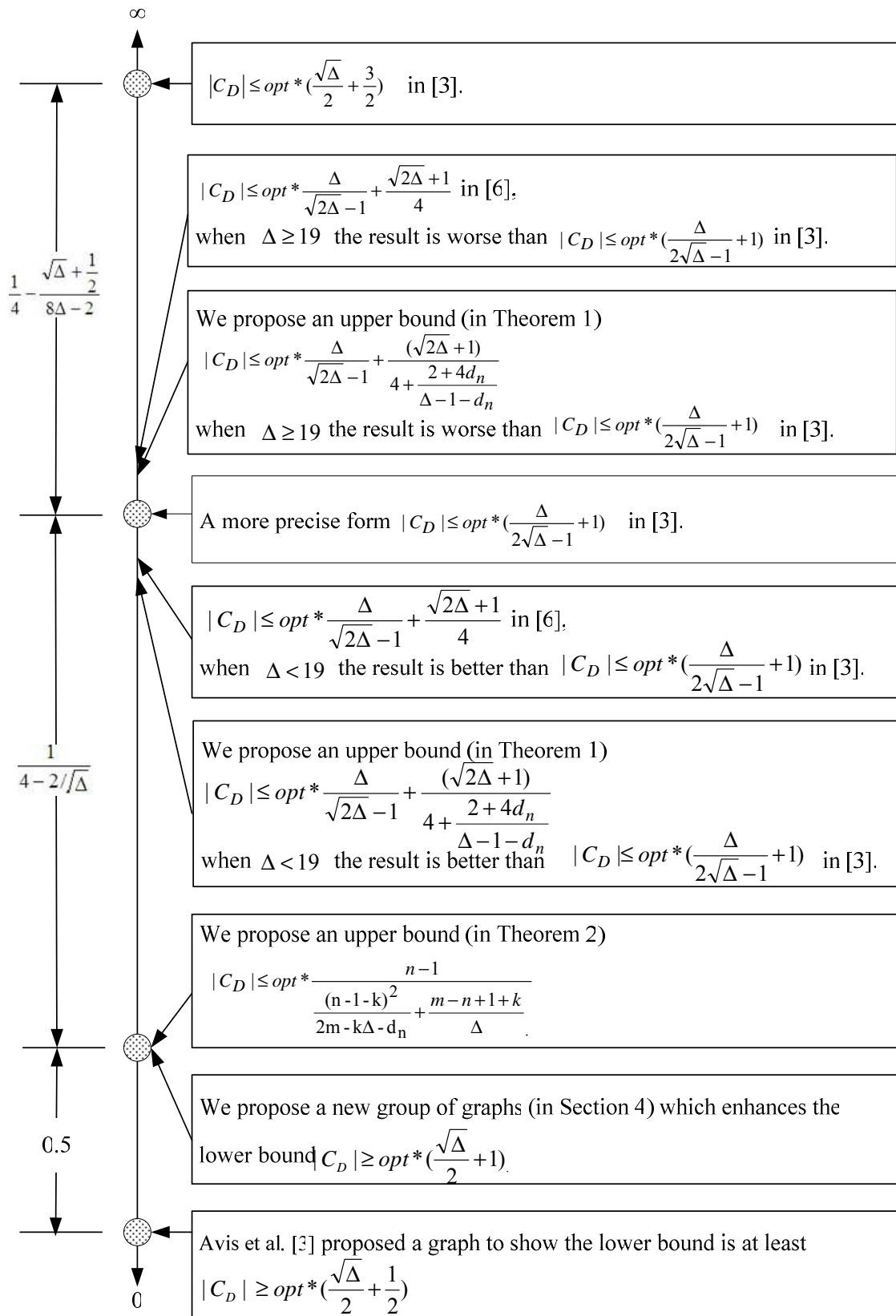$$|C_D| \geq opt * (\frac{\sqrt{\Delta}}{2} + \frac{1}{2})$$

0

Figure 7. The bounds of ListLeft.

Table 1. Comparison of the results with previous work.

Researcher	Main ideas	The results of the bound equation
Avis et al. [3]	$opt \geq \sum y_e = \sum\limits_{i=1}^{s}\frac{1}{d_i}+\frac{1}{\Delta}(m-s)$ $\geq \frac{s^2}{m}+\frac{m-s}{\Delta}=\frac{s^2}{m}+\frac{m}{\Delta}-\frac{s}{\Delta}$	$\mid C_D \mid \leq opt*(\frac{\Delta}{2\sqrt{\Delta}-1}+1)$
Iida Hiroshi [5]	$opt \geq \sum y_e = \sum\limits_{i=1}^{p}\frac{1}{d_i}+\sum\limits_{i=1}^{s}\frac{1}{d_i}+\frac{m-C_D}{\Delta}$ $\geq \frac{p^2}{m-1+\Delta}+\frac{s^2}{m}+\frac{m-C_D}{\Delta}$	$\mid C_D \mid \leq opt*\frac{\Delta}{\sqrt{2\Delta}-1}+\frac{\sqrt{2\Delta}+1}{4}$
Iida Hiroshi [6]	$opt \geq \sum y_e = \sum\limits_{i=1}^{C_D}\frac{1}{d_i}+\frac{1}{\Delta}(m-C_D)$ $\geq \frac{C_D^2}{2m-q}+\frac{2(m-C_D)}{2\Delta}$ where $q \geq d_n$	$\mid C_D \mid \leq opt*\frac{\Delta}{\sqrt{2\Delta}-1}-\frac{q}{2\sqrt{2\Delta}-2}$
Our result in Theorem 1	$opt \geq \sum y_e = \sum\limits_{i=1}^{p}\frac{1}{d_i} + \sum\limits_{i=1}^{s}\frac{1}{d_i} + \frac{m-C_D}{d_1}$	$\mid C_D \mid \leq opt*\frac{\Delta}{\sqrt{2\Delta}-1}+\frac{(\sqrt{2\Delta}+1)}{4+\frac{2+4d_n}{\Delta-1-d_n}}$
Our result in Theorem 2	$opt \geq \Sigma y_e = \frac{1}{d_1}+\frac{1}{d_2}+...+\frac{1}{d_{n-1}}+\frac{m-(n-1)}{d_1}$ $\geq \frac{(n-1-k)^2}{2m-k\Delta-d_n}+\frac{m-(n-1-k)}{\Delta}$	$\mid C_D \mid \leq opt*\frac{n-1}{\frac{(n-1-k)^2}{2m-k\Delta-d_n}+\frac{m-n+1+k}{\Delta}}$

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