國立臺灣師範大學資訊工程學系
九十九學年度第二學期
博士班資格考

考試科目：線性代數
總分一百分
請在答案卷作答，在題目卷上作答不予計分

1. (10pts) Please show that T is a linear transformation by finding a matrix that implements the mapping, where x_1, x_2, x_3, and x_4 are entries in vectors.
   $T(x_1, x_2, x_3, x_4) = (0, x_1+ x_2, x_2+x_3, x_3+ x_4, 2x_2-3x_1)$.

2. (10pts) Given the following vectors v_1, v_2 and v_3.
   (a) Please find the value(s) of h for which the vector v_3 is in Span$\{v_1, v_2\}$.
   (b) Please find the value(s) of h for which the vectors v_1, v_2 and v_3 are linear independent.

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \quad v_2 = \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix} \quad v_3 = \begin{bmatrix} -1 \\ 5 \\ h \end{bmatrix}$$

3. (10pts) Given the following matrix A and an echelon form of A.

$$A = \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 6 & -5 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   (a) Please find a basis for the column space of A and a basis for the Null space of A (denoted as Null A).
   (b) What's the value of the rank of A. What's the dimension of Null A?

4. (10pts) Given the following matrix B. Please find $\det(((B^4)B^T)^{-1})$.

$$B = \begin{bmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{bmatrix}$$

5. (10 pts) Let $P=\begin{bmatrix}5 & 3\\ 6 & 4\end{bmatrix}$, $v_1=\begin{bmatrix}1\\ -3\end{bmatrix}$, and $v_2=\begin{bmatrix}-2\\ 4\end{bmatrix}$.

Please find a basis $\{u_1, u_2\}$ for R^2 such that P is the change-of coordinates matrix from the basis $\{u_1, u_2\}$ to the basis $\{v_1, v_2\}$.

6. (5 pts) Let $C = [\ c_{ij}\ ]$ be an n by n matrix and λ be an eigenvalue of C. Determine the relationship ($>$, $\geq$, $<$, $\leq$ or $=$) between $|\lambda|$ and $\|C\|$, where $\|C\| = (\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^2)^{1/2}$.

7. (10 pts) Solve the initial value problem:

$$\begin{cases} x_1'(t) = x_1(t) + x_2(t) + 6e^{2t} \\ x_2'(t) = x_1(t) + x_2(t) + 2e^{2t} \end{cases}, \quad \text{where } x_1(0) = 6, \text{ and } x_2(0) = 0.$$

8. (15 pts) Consider the over-constrained linear system $U\boldsymbol{x} = \boldsymbol{y}$, where

$$U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1q} \\ u_{21} & u_{22} & \cdots & u_{2q} \\ \cdots & \cdots & \cdots & \cdots \\ u_{p1} & u_{p2} & \cdots & u_{pq} \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_q \end{pmatrix} \text{ and } \boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ \cdots \\ y_p \end{pmatrix}, (p > q).$$

The least-squares solution of the above system is given by $\boldsymbol{x} = (U^T U)^{-1} U^T \boldsymbol{y}$. Provide the unit (i.e., $\|\boldsymbol{x}\| = 1$) least squares solution of the homogeneous system $U\boldsymbol{x} = \boldsymbol{0}$.

9. (20 pts) Let $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ so that quadratic form $P(\boldsymbol{x}) = 3x_1^2 + 3x_2^2 + 2x_1x_2$ can be expressed in a compact vector-matrix form as $P(\boldsymbol{x}) = \boldsymbol{x}^T A\boldsymbol{x}$, where $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

Suppose orthonormal matrix H relates variables $\boldsymbol{x}$ and $\boldsymbol{y}$ by $\boldsymbol{x} = H\boldsymbol{y}$, through which quadratic form $P(\boldsymbol{x})$ is transformed into another quadratic form $Q(\boldsymbol{y}) = ay_1^2 + by_2^2$ with no cross-product term y_1y_2 in which a and b are constants.

(a) Find matrix H, and constants a and b.

(b) Find the maximum value of $P(\boldsymbol{x})$ subject to $\|\boldsymbol{x}\| = 1$.

(c) Find a unit vector at which the maximum is attained.